

# MAXIMAL FUNCTION CHARACTERIZATIONS FOR HARDY SPACES ASSOCIATED TO NONNEGATIVE SELF-ADJOINT OPERATORS ON SPACES OF HOMOGENEOUS TYPE

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**ABSTRACT.** Let  $X$  be a metric measure space with a doubling measure and  $L$  be a nonnegative self-adjoint operator acting on  $L^2(X)$ . Assume that  $L$  generates an analytic semigroup  $e^{-tL}$  whose kernels  $p_t(x, y)$  satisfy Gaussian upper bounds but without any assumptions on the regularity of space variables  $x$  and  $y$ . In this article we continue a study in [21] to give an atomic decomposition for the Hardy spaces  $H_{L, \max}^p(X)$  in terms of the nontangential maximal function associated with the heat semigroup of  $L$ , and hence we establish characterizations of Hardy spaces associated to an operator  $L$ , via an atomic decomposition or the nontangential maximal function. We also obtain an equivalence of  $H_{L, \max}^p(X)$  in terms of the radial maximal function.

## 1. INTRODUCTION

Our goal in this paper is to continue a study in [21] to establish the equivalence of the maximal and atomic Hardy spaces on spaces of homogeneous type, associated to nonnegative self-adjoint operators whose heat kernel has Gaussian upper bounds. For the theory of Hardy spaces associated to operators, it has attracted a lot of attention in the last decades, and has been a very active research topic in harmonic analysis – see for example, [1, 2, 3, 7, 10, 11, 12, 13, 15, 16, 17, 18, 21, 23, 24].

Let  $(X, d, \mu)$  be a metric measure space endowed with a distance  $d$  and a nonnegative Borel doubling measure  $\mu$  on  $X$  ([9]). Recall that a measure  $\mu$  is doubling provided that there exists a constant  $C > 0$  such that for all  $x \in X$  and for all  $r > 0$ ,

$$(1.1) \quad V(x, 2r) \leq CV(x, r)$$

where  $V(x, r) = \mu(B(x, r))$ , the volume of the open ball  $B = B(x, r) := \{y \in X : d(y, x) < r\}$ . Note that the doubling property implies the following strong homogeneity property,

$$(1.2) \quad V(x, \lambda r) \leq C\lambda^n V(x, r)$$

for some  $C, n > 0$  uniformly for all  $\lambda \geq 1, r > 0$  and  $x \in X$ . In Euclidean space with Lebesgue measure, the parameter  $n$  corresponds to the dimension of the space, but in our more abstract setting, the optimal  $n$  need not even be an integer. There also exists  $C > 0$  so that

$$(1.3) \quad V(y, r) \leq C \left(1 + \frac{d(x, y)}{r}\right)^n V(x, r)$$

uniformly for all  $x, y \in X$  and  $r > 0$ . Indeed, property (1.3) is a direct consequence of the triangle inequality for the metric  $d$  and the strong homogeneity property (1.2).

The following will be assumed throughout the article unless otherwise specified:

**(H1)**  $L$  is a non-negative self-adjoint operator on  $L^2(X)$ ;

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**(H2)** The kernel of  $e^{-tL}$ , denoted by  $p_t(x, y)$ , is a measurable function on  $X \times X$  and satisfies a Gaussian upper bound, that is

$$(GE) \quad |p_t(x, y)| \leq C \frac{1}{V(x, \sqrt{t})} \exp\left(-\frac{d(x, y)^2}{ct}\right)$$

for all  $t > 0$ , and  $x, y \in X$ , where  $C$  and  $c$  are positive constants.

We now recall the notion of a  $(p, q, M)$ -atom associated to an operator  $L$  ([2, 11, 15]).

**Definition 1.1.** Given  $0 < p \leq 1 \leq q \leq \infty$ ,  $p < q$  and  $M \in \mathbb{N}$ , a function  $a \in L^2(X)$  is called a  $(p, q, M)$ -atom associated to the operator  $L$  if there exist a function  $b \in \mathcal{D}(L^M)$  and a ball  $B \subset X$  such that

- (i)  $a = L^M b$ ;
- (ii)  $\text{supp } L^k b \subset B$ ,  $k = 0, 1, \dots, M$ ;
- (iii)  $\|(r_B^2 L)^k b\|_{L^q(X)} \leq r_B^{2M} V(B)^{1/q-1/p}$ ,  $k = 0, 1, \dots, M$ .

The atomic Hardy space  $H_{L,at,q,M}^p(X)$  is defined as follows.

**Definition 1.2.** We will say that  $f = \sum \lambda_j a_j$  is an atomic  $(p, q, M)$ -representation (of  $f$ ) if  $\{\lambda_j\}_{j=0}^\infty \in \ell^p$ , each  $a_j$  is a  $(p, q, M)$ -atom, and the sum converges in  $L^2(X)$ . Set

$$\mathbb{H}_{L,at,q,M}^p(X) := \left\{ f : f \text{ has an atomic } (p, q, M)\text{-representation} \right\},$$

with the norm  $\|f\|_{\mathbb{H}_{L,at,q,M}^p(X)}$  given by

$$\inf \left\{ \left( \sum_{j=0}^\infty |\lambda_j|^p \right)^{1/p} : f = \sum_{j=0}^\infty \lambda_j a_j \text{ is an atomic } (p, q, M)\text{-representation} \right\}.$$

The space  $H_{L,at,q,M}^p(X)$  is then defined as the completion of  $\mathbb{H}_{L,at,q,M}^p(X)$  with respect to this norm.

Given a function  $f \in L^2(X)$ , consider the following non-tangential maximal function associated to the heat semigroup generated by the operator  $L$ ,

$$(1.4) \quad f_L^*(x) =: \sup_{d(x,y) < t} |e^{-t^2 L} f(y)|.$$

We may define the spaces  $H_{L,max}^p(X)$ ,  $0 < p \leq 1$  as the completion of  $\{L^2(X) : \|f_L^*\|_{L^p(X)} < \infty\}$  with respect to  $L^p$ -norm of the non-tangential maximal function; i.e.,

$$(1.5) \quad \|f\|_{H_{L,max}^p(X)} := \|f_L^*\|_{L^p(X)}$$

It can be verified (see [15, 11, 18]) that for all  $q > p$  with  $1 \leq q \leq \infty$  and every number  $M > \frac{n}{2}(\frac{1}{p}-1)$ , any  $(p, q, M)$ -atom  $a$  is in  $H_{L,max}^p(X)$  and so the following continuous inclusion holds:

$$(1.6) \quad H_{L,at,q,M}^p(X) \subseteq H_{L,max}^p(X).$$

A natural question is to show the following continuous inclusion  $H_{L,max}^p(X) \subseteq H_{L,at,q,M}^p(X)$ . In the case of  $X = \mathbb{R}^n$ , it is known that the inclusion  $H_{L,max}^p(\mathbb{R}^n) \subseteq H_{L,at,q,M}^p(\mathbb{R}^n)$  holds for certain operators including Schrödinger operators with nonnegative potentials and second order divergence form elliptic operators via particular PDE technique (see for example, [13, 14, 15, 16]). Very recently, the authors of this article have made a reformulation and modification of a technique due to A. Calderón [5] to obtain an atomic decomposition directly from  $H_{L,max}^p(\mathbb{R}^n)$ . Precisely, under the assumptions **(H1)** and **(H2)** of the operator  $L$  but without any assumptions on the regularity of

$p_t(x, y)$  of space variables  $x$  and  $y$ , we have that  $H_{L, \max}^p(\mathbb{R}^n) \subseteq H_{L, \text{at}, q, M}^p(\mathbb{R}^n)$  for  $0 < p \leq 1 \leq q \leq \infty$  with  $q > p$ , and all integers  $M > \frac{n}{2}(\frac{1}{p} - 1)$ , and hence by (1.6),

$$H_{L, \max}^p(\mathbb{R}^n) \simeq H_{L, \text{at}, q, M}^p(\mathbb{R}^n).$$

We point out that in [5], a decomposition of the function  $F(x, t) = f * \varphi_t(x)$  associated with the distribution  $f$  was given, and convolution operation of the function  $F$  played an important role in the proof. In [21, Theorem 1.4], no analogue of convolution operation of the function  $t^2 L e^{-t^2 L} f(x)$ , the proof depends critically on the geometry of  $\mathbb{R}^n$  to use oblique cylinders of  $\mathbb{R}_+^{n+1}$ : for every cube  $Q$  of  $\mathbb{R}^n$  and for  $\bar{e} = (1, \dots, 1) \in \mathbb{R}^n$

$$\tilde{Q} := \{(y, t) \in \mathbb{R}_+^{n+1} : y + 3t\bar{e} \in Q\},$$

in place of vertical cylinders in Calderón's construction in [5]. However, "oblique cylinders" do not exist on spaces of homogeneous type and hence it is not trivial to generalize the method in [21] to the case of spaces of homogeneous type. So, we may ask the following question:

**Question 1.** Is it possible to show an inclusion  $H_{L, \max}^p(X) \subseteq H_{L, \text{at}, q, M}^p(X)$  on space of homogeneous type  $X$ ?

Next we consider the Hardy space  $H_{L, \max}^p(X)$  in terms of the radial maximal function. Given an operator  $L$  satisfying (H1)-(H2), we may define the spaces  $H_{L, \text{rad}}^p(X)$ ,  $0 < p \leq 1$  as the completion of  $\{f \in L^2(X) : \|f_L^+\|_{L^p(X)} < \infty\}$  with respect to  $L^p$ -norm of the radial maximal function; i.e.,

$$(1.7) \quad \|f\|_{H_{L, \text{rad}}^p(X)} := \|f_L^+\|_{L^p(X)} := \left\| \sup_{t>0} |e^{-t^2 L} f| \right\|_{L^p(X)}.$$

Fix  $0 < p \leq 1$ . For all  $q > p$  with  $1 \leq q \leq \infty$  and for all integers  $M > \frac{n}{2}(\frac{1}{p} - 1)$ , the following continuous inclusion holds:

$$(1.8) \quad H_{L, \text{at}, q, M}^p(X) \subseteq H_{L, \max}^p(X) \subseteq H_{L, \text{rad}}^p(X)$$

by (1.4)-(1.7). In the case of  $X = \mathbb{R}^n$ , it is known that the inclusion  $H_{L, \text{rad}}^p(\mathbb{R}^n) \subseteq H_{L, \max}^p(\mathbb{R}^n)$  holds for certain operators including Schrödinger operators with nonnegative potentials and second order divergence form elliptic operators via particular PDE technique (see example, [13, 14, 15]). We may ask the following question:

**Question 2.** Is it possible to show an inclusion  $H_{L, \text{rad}}^p(X) \subseteq H_{L, \max}^p(X)$  assuming merely that an operator  $L$  satisfies (H1)-(H2)?

The aim of this article is give an affirmative answer to Questions 1 and 2 above to establish the equivalent characterizations of Hardy space associated to an operator  $L$  satisfies (H1) and (H2) on spaces of homogeneous type  $X$ , including an atomic decomposition, the nontangential maximal functions and the radial maximal functions. Throughout the article, we always assume that  $\mu(X) = \infty$  and  $\mu(\{x\}) = 0$  for all  $x \in X$ .

**Theorem 1.3.** Let  $(X, d, \mu)$  be as in (1.1) and (1.2). Suppose that an operator  $L$  satisfies (H1) and (H2). Fix  $0 < p \leq 1$ . For all  $q > p$  with  $1 \leq q \leq \infty$  and for all integers  $M > \frac{n}{2}(\frac{1}{p} - 1)$ , we have that

- (i)  $H_{L, \text{rad}}^p(X) \subseteq H_{L, \max}^p(X)$ ;
- (ii)  $H_{L, \max}^p(X) \subseteq H_{L, \text{at}, q, M}^p(X)$ .

Hence by (1.8),

$$H_{L, \text{at}, q, M}^p(X) \simeq H_{L, \max}^p(X) \simeq H_{L, \text{rad}}^p(X).$$

The layout of the article is as follows. In Section 2 we recall some basic properties of heat kernels and finite propagation speed for the wave equation, and build the necessary kernel estimates for functions of an operator, which is useful in the proof of Theorem 1.3. In Section 3 we show (i) of Theorem 1.3 to obtain an equivalence of Hardy spaces on spaces of homogeneous type, in terms of the nontangential and radial maximal functions. In Section 4 we will show our main result, (ii) of Theorem 1.3. A crucial idea in the proof is to make a modification of [5, 21] to decompose  $X \times (0, \infty)$  into type of “cones” in place of “vertical cylinders” of  $\mathbb{R}_+^{n+1}$  in [5] or “oblique cylinders” of  $\mathbb{R}_+^{n+1}$  in [21] (see (4.6) below). This leads us to obtain characterizations of Hardy spaces associated to an operator  $L$  on spaces of homogeneous type, via an atomic decomposition or the nontangential maximal function.

Throughout, the letter “ $c$ ” and “ $C$ ” will denote (possibly different) constants that are independent of the essential variables.

## 2. PRELIMINARIES

Recall that, if  $L$  is a nonnegative, self-adjoint operator on  $L^2(X)$ , and  $E_L(\lambda)$  denotes a spectral decomposition associated with  $L$ , then for every bounded Borel function  $F : [0, \infty) \rightarrow \mathbb{C}$ , one defines the operator  $F(L) : L^2(X) \rightarrow L^2(X)$  by the formula

$$(2.1) \quad F(L) := \int_0^\infty F(\lambda) dE_L(\lambda).$$

In particular, the operator  $\cos(t\sqrt{L})$  is then well-defined on  $L^2(X)$ . Moreover, it follows from Theorem 3.4 of [8] that the integral kernel  $K_{\cos(t\sqrt{L})}$  of  $\cos(t\sqrt{L})$  satisfies

$$(2.2) \quad \text{supp} K_{\cos(t\sqrt{L})} \subseteq \{(x, y) \in X \times X : d(x, y) \leq t\}.$$

By the Fourier inversion formula, whenever  $F$  is an even bounded Borel function with the Fourier transform of  $F$ ,  $\hat{F} \in L^1(\mathbb{R})$ , we can write  $F(\sqrt{L})$  in terms of  $\cos(t\sqrt{L})$ . Concretely, by recalling (2.1) we have

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^\infty \hat{F}(t) \cos(t\sqrt{L}) dt,$$

which, when combined with (2.2), gives

$$(2.3) \quad K_{F(\sqrt{L})}(x, y) = (2\pi)^{-1} \int_{|t| \geq d(x, y)} \hat{F}(t) K_{\cos(t\sqrt{L})}(x, y) dt.$$

This property leads us the following result (see [15, Lemma 3.5]): For every even function  $\varphi \in C_0^\infty(\mathbb{R})$  with  $\text{supp } \varphi \subset (-1, 1)$ ,

$$(2.4) \quad \text{supp} K_{\Phi(t\sqrt{L})} \subseteq \{(x, y) \in X \times X : d(x, y) \leq t\},$$

where  $\Phi$  denotes the Fourier transform of  $\varphi$ .

**Lemma 2.1.** *Assume that an operator  $L$  satisfies (H1)-(H2).*

- (i) *Let  $\varphi \in \mathcal{S}(\mathbb{R})$  be an even function. Then for every  $\beta > 0$ , there exists a positive constant  $C = C(n, \beta, \varphi)$  such that the kernel  $K_{\varphi(t\sqrt{L})}(x, y)$  of  $\varphi(t\sqrt{L})$  satisfies*

$$(2.5) \quad |K_{\varphi(t\sqrt{L})}(x, y)| \leq C \frac{1}{\max(V(x, t), V(y, t))} \left(1 + \frac{d(x, y)}{t}\right)^{-n-\beta}$$

*for all  $t > 0$  and  $x, y \in X$ .*

(ii) Let  $\psi_i \in \mathcal{S}(\mathbb{R})$  be even functions,  $\psi_i(0) = 0, i = 1, 2$ . Then for every  $\beta > 0$ , there exists a positive constant  $C = C(n, \beta, \psi_1, \psi_2)$  such that the kernel  $K_{\psi_1(s\sqrt{L})\psi_2(t\sqrt{L})}(x, y)$  of  $\psi_1(s\sqrt{L})\psi_2(t\sqrt{L})$  satisfies

(2.6)

$$|K_{\psi_1(s\sqrt{L})\psi_2(t\sqrt{L})}(x, y)| \leq C \min\left(\frac{s}{t}, \frac{t}{s}\right) \frac{1}{\max(V(x, \max(s, t)), V(y, \max(s, t)))} \left(1 + \frac{d(x, y)}{\max(s, t)}\right)^{-n-\beta}$$

for all  $t > 0$  and  $x, y \in X$ .

*Proof.* The proof of (i) and (ii) is similar to that of [4, Lemma 2.3] and [21, Lemma 2.3] on the Euclidean spaces  $\mathbb{R}^n$ , respectively. We omit the detail here.  $\square$

Let  $F(y, t)$  be a  $\mu$ -measurable function of  $X \times (0, +\infty)$ . For  $\alpha > 0$ , set  $F_\alpha^*(x) = \sup_{d(x, y) < \alpha t} |F(y, t)|$ .

With the notation above, we have the following result.

**Lemma 2.2.** For any  $p > 0$  and  $0 < \alpha_2 \leq \alpha_1$ ,

$$\|F_{\alpha_1}^*\|_{L^p(X)} \leq C \left(1 + \frac{2\alpha_1}{\alpha_2}\right)^{n/p} \|F_{\alpha_2}^*\|_{L^p(X)},$$

where  $C = C(p, n)$  is independent of  $\alpha_1, \alpha_2$  and  $F$ .

*Proof.* The proof of Lemma 2.2 is standard (see for instance, [6, Theorem 2.3] for the case of  $X = \mathbb{R}^n$ ). We write

$$\|F_{\alpha_1}^*\|_{L^p(X)}^p = p \int_0^\infty \lambda^{p-1} \mu\{x \in X : F_{\alpha_1}^*(x) > \lambda\} d\lambda.$$

Observe that

$$(2.7) \quad \{x \in X : F_{\alpha_1}^*(x) > \lambda\} \subset \left\{x \in X : \mathcal{M}(\chi_E)(x) > C^{-2} \left(1 + \frac{2\alpha_1}{\alpha_2}\right)^{-n}\right\},$$

where  $E := \{x \in X : F_{\alpha_2}^*(x) > \lambda\}$  and  $\mathcal{M}$  denotes the Hardy–Littlewood maximal function. Indeed, if  $F_{\alpha_1}^*(x_0) > \lambda$ , then there exist  $y_0 \in X$  and  $t_0 > 0$  such that  $d(x_0, y_0) < \alpha_1 t_0$  and  $|F(y_0, t_0)| > \lambda$ . Hence,  $B(y_0, \alpha_2 t_0) \subset E$ . It follows that

$$\frac{\mu(B(x_0, (\alpha_1 + \alpha_2)t_0) \cap E)}{V(x_0, (\alpha_1 + \alpha_2)t_0)} \geq \frac{V(y_0, \alpha_2 t_0)}{V(x_0, (\alpha_1 + \alpha_2)t_0)}.$$

By (1.2) and (1.3),

$$\begin{aligned} V(x_0, (\alpha_1 + \alpha_2)t_0) &\leq CV(y_0, (\alpha_1 + \alpha_2)t_0) \left(1 + \frac{d(x_0, y_0)}{(\alpha_1 + \alpha_2)t_0}\right)^n \\ &\leq C^2 V(y_0, \alpha_2 t_0) \left(\frac{\alpha_1 + \alpha_2}{\alpha_2}\right)^n \left(1 + \frac{\alpha_1}{\alpha_1 + \alpha_2}\right)^n \\ &\leq C^2 V(y_0, \alpha_2 t_0) \left(1 + \frac{2\alpha_1}{\alpha_2}\right)^n, \end{aligned}$$

which gives

$$\frac{\mu(B(x_0, (\alpha_1 + \alpha_2)t_0) \cap E)}{V(x_0, (\alpha_1 + \alpha_2)t_0)} \geq C^{-2} \left(1 + \frac{2\alpha_1}{\alpha_2}\right)^{-n}.$$

This proves (2.7). By the weak (1,1) boundedness of Hardy–Littlewood maximal function, we obtain the proof of Lemma 2.2.  $\square$

**Proposition 2.3.** *Let  $0 < p \leq 1$ . Suppose that an operator  $L$  satisfies **(H1)** and **(H2)**. Let  $\varphi_i \in \mathcal{S}(\mathbb{R})$  be even functions with  $\varphi_i(0) = 1$  and  $\alpha_i > 0, i = 1, 2$ . Then there exists a constant  $C = C(n, \varphi_1, \varphi_2, \alpha_1, \alpha_2)$  such that for every  $f \in L^2(X)$ , the functions  $\varphi_{i,L,\alpha}^* f = \sup_{d(x,y) < \alpha t} |\varphi_i(t\sqrt{L})f(y)|, i = 1, 2$ , satisfy*

$$(2.8) \quad \|\varphi_{1,L,\alpha_1}^* f\|_{L^p(X)} \leq C \|\varphi_{2,L,\alpha_2}^* f\|_{L^p(X)}.$$

As a consequence, for any even function  $\varphi \in \mathcal{S}(\mathbb{R})$  with  $\varphi(0) = 1$  and  $\alpha > 0$ ,

$$C^{-1} \|f_L^*\|_{L^p(X)} \leq \|\varphi_{L,\alpha}^* f\|_{L^p(X)} \leq C \|f_L^*\|_{L^p(X)}.$$

*Proof.* The argument is similar to that of [21, Proposition 3.1] with minor modifications. We give a brief argument of this proof for completeness and convenience for the reader.

For any  $0 < \alpha_2 \leq \alpha_1$ , we apply Lemma 2.2 to have that

$$\|\varphi_{L,\alpha_1}^* f\|_{L^p(X)} \leq C(p, n) \left(1 + \frac{2\alpha_1}{\alpha_2}\right)^{n/p} \|\varphi_{L,\alpha_2}^* f\|_{L^p(X)}$$

for any  $\varphi \in \mathcal{S}(\mathbb{R})$ . Now, we let  $\psi(x) := \varphi_1(x) - \varphi_2(x)$ , and then the proof of (4.1) reduces to show that

$$(2.9) \quad \|\psi_{L,1}^* f\|_{L^p(X)} \leq C \|\varphi_{2,L,1}^* f\|_{L^p(X)}.$$

Let us show (2.9). Let  $\varphi \in C_0^\infty(\mathbb{R})$  be even,  $\text{supp } \varphi \subset (-1, 1)$ . Let  $\Phi$  denote the Fourier transform of  $\varphi$  and set  $\Psi(x) := x^{2\kappa} \Phi(x)$  and  $2\kappa > (n+1)/p$ . By the spectral theory ([23]), we have

$$f = C_{\Psi, \varphi_2} \int_0^\infty \Psi(s\sqrt{L}) \varphi_2(s\sqrt{L}) f \frac{ds}{s}.$$

Therefore,

$$\psi(t\sqrt{L})f(x) = C \int_0^\infty (\psi(t\sqrt{L})\Psi(s\sqrt{L})) \varphi_2(s\sqrt{L})f(x) \frac{ds}{s}.$$

Let us denote the kernel of  $\psi(t\sqrt{L})\Psi(s\sqrt{L})$  by  $K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x, y)$ . For every  $\lambda \in (\frac{n}{p}, 2\kappa)$ , we write

$$(2.10) \quad \begin{aligned} & \sup_{d(x,y) < t} |\psi(t\sqrt{L})f(y)| \\ &= C \sup_{d(x,y) < t} \left| \int_0^\infty \int_X K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(y, z) \varphi_2(s\sqrt{L})f(z) d\mu(z) \frac{ds}{s} \right| \\ &\leq \sup_{z,s} |\varphi_2(s\sqrt{L})f(z)| \left(1 + \frac{d(x,z)}{s}\right)^{-\lambda} \sup_{d(x,y) < t} \int_0^\infty \int_X |K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(y, z)| \left(1 + \frac{d(x,z)}{s}\right)^\lambda d\mu(z) \frac{ds}{s}. \end{aligned}$$

By (ii) of Lemma 2.1, it follows that for  $\eta \in (\lambda, 2\kappa)$ ,

$$|K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(y, z)| \leq C \min\left(\left(\frac{s}{t}\right)^{2\kappa}, \left(\frac{t}{s}\right)^2\right) \frac{1}{\left(1 + \frac{d(y,z)}{\max(s,t)}\right)^{n+\eta}} \frac{1}{V(y, \max(s, t))}.$$

For any  $d(x, y) < t$ , one can compute

$$\int_X \frac{1}{V(y, \max(s, t))} \frac{1}{\left(1 + \frac{d(y,z)}{\max(s,t)}\right)^{n+\eta}} \left(1 + \frac{d(x,z)}{s}\right)^\lambda d\mu(z) \leq C \max\left(1, \left(\frac{t}{s}\right)^\eta\right),$$

which implies,

$$\begin{aligned} \int_X |K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(y, z)| \left(1 + \frac{d(x, z)}{s}\right)^\lambda d\mu(z) &\leq C \min\left(\left(\frac{s}{t}\right)^{2\kappa}, \left(\frac{t}{s}\right)^2\right) \max\left(1, \left(\frac{t}{s}\right)^\eta\right) \\ &\leq C \min\left(\left(\frac{s}{t}\right)^{2\kappa-\eta}, \left(\frac{t}{s}\right)^2\right), \end{aligned}$$

for any  $d(x, y) < t$ . Hence

$$\sup_{d(x, y) < t} \int_0^\infty \int_X |K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(y, z)| \left(1 + \frac{d(x, z)}{s}\right)^\lambda d\mu(z) \frac{ds}{s} \leq C \int_0^\infty \min\left(\left(\frac{s}{t}\right)^{2\kappa-\eta}, \left(\frac{t}{s}\right)^2\right) \frac{ds}{s} \leq C.$$

This, in combination with (2.10) and the condition  $\lambda \in (\frac{n}{p}, 2\kappa)$ , implies

$$\begin{aligned} \|\psi_{L,1}^* f\|_{L^p(X)} &= \left\| \sup_{d(x,y) < t} |\psi(t\sqrt{L})f(y)| \right\|_{L_x^p(X)} \leq C \left\| \sup_{z,s} |\varphi_2(s\sqrt{L})f(z)| \left(1 + \frac{d(x, z)}{s}\right)^{-\lambda} \right\|_{L_x^p(X)} \\ &\leq C \left\| \sup_{d(x,y) < t} |\varphi_2(t\sqrt{L})f(y)| \right\|_{L_x^p(X)} \\ &= C \|\varphi_{2,L,1}^* f\|_{L^p(X)}, \end{aligned}$$

where the second inequality above can be proved easily by combining Lemma 2.2 and the argument of [6, Theorem 2.4]. This completes the proof of Proposition 2.3.  $\square$

### 3. EQUIVALENCE OF HARDY SPACES $H_{L,max}^p(X)$ AND $H_{L,rad}^p(X)$

Assume that the metric measure space  $X$  satisfies the doubling conditions (1.1) and (1.2) with exponent  $n$ . In this section we will show (i) of Theorem 1.3 to obtain an equivalence of Hardy spaces  $H_{L,rad}^p(X)$  and  $H_{L,max}^p(X)$ . By (1.4) and (1.7), we have

$$(3.1) \quad H_{L,max}^p(X) \subseteq H_{L,rad}^p(X).$$

Now for every even functions  $\varphi \in \mathcal{S}(\mathbb{R})$  with  $\varphi(0) = 1$ , and for every  $f \in L^2(X)$  we define

$$\varphi_L^*(f)(x) = \sup_{d(x,y) < t} |\varphi(t\sqrt{L})f(y)|$$

and

$$\varphi_L^+(f)(x) = \sup_{t>0} |\varphi(t\sqrt{L})f(x)|$$

Then we have the the following result.

**Theorem 3.1.** *Let  $(X, d, \mu)$  be as in (1.1) and (1.2). Suppose that an operator  $L$  satisfies (H1) and (H2). Fix  $0 < p \leq 1$ . Then there exists a constant  $C = C(p, \varphi) > 0$  such that for every  $f \in L^2(X)$ ,*

$$(3.2) \quad \|\varphi_L^*(f)\|_{L^p(X)} \leq C \|\varphi_L^+(f)\|_{L^p(X)},$$

and hence by (3.1)

$$H_{L,max}^p(X) \simeq H_{L,rad}^p(X).$$

*Proof.* For every  $N > 0$ , we define

$$M_{L,\varphi,N}^{**}(f)(x) := \sup_{y \in X, s>0} \frac{|\varphi(s\sqrt{L})f|}{\left(1 + \frac{d(x,y)}{s}\right)^N}.$$



By (1.4), we have

$$(3.3) \quad \varphi_L^*(f)(x) \leq 2^N M_{L,\varphi,N}^{**}(f)(x).$$

We now claim that if  $0 < \theta < 1$  and  $N\theta > 2n$ , then there exists  $C = C(p, \varphi, N, \theta) > 0$  such that for every  $f \in L^2(X)$ ,

$$(3.4) \quad M_{L,\varphi,N}^{**}(f)(x) \leq C \left[ \mathcal{M}((\varphi_L^+(f))^\theta)(x) \right]^{1/\theta} \text{ a.e. } x \in X.$$

If the claim is proved, then we can choose  $N = 2(n+1)/p$  and  $\theta = \frac{(2n+1)p}{2(n+1)}$  and apply the  $L^r$  ( $r > 1$ ) boundedness of Hardy–Littlewood maximal operator to obtain that for any  $f \in L^2(X)$

$$\|M_{L,\varphi,N}^{**}(f)\|_{L^p(X)} \leq C \left\| \left[ \mathcal{M}((\varphi_L^+(f))^\theta)(x) \right]^{1/\theta} \right\|_{L^p(X)} \leq C \|\varphi_L^+(f)\|_{L^p(X)},$$

which, together with (3.3), yields (3.2).

It remains to prove (3.4). Let  $\varphi \in C_0^\infty(\mathbb{R})$  be even,  $\text{supp } \varphi \subset (-1, 1)$ . Let  $\Phi$  denote the Fourier transform of  $\varphi$  and set  $\Psi(x) := x^{2\kappa}\Phi(x)$ ,  $x \in \mathbb{R}$  and  $\kappa > N/2$ . For every  $f \in L^2(X)$  one can write

$$(3.5) \quad f = \lim_{\epsilon \rightarrow 0} c_{\Psi,\varphi} \int_{\epsilon}^{1/\epsilon} \Psi(t\sqrt{L})\varphi(t\sqrt{L})f \frac{dt}{t}$$

with the integral converging in  $L^2(X)$ .

Set

$$\eta(x) := c_{\Psi,\varphi} \int_1^\infty \Psi(tx)\varphi(tx)\frac{dt}{t} = c_{\Psi,\varphi} \int_x^\infty \Psi(y)\varphi(y)\frac{dy}{y}, \quad x \neq 0$$

with  $\eta(0) = 1$ . It follows that  $\eta \in \mathcal{S}(\mathbb{R})$  is an even function. By the spectral theory ([25]) again, one can write, for any  $s > 0$ ,

$$(3.6) \quad \eta(s\sqrt{L})f = c_{\Psi,\varphi} \int_s^\infty \Psi(t\sqrt{L})\varphi(t\sqrt{L})f \frac{dt}{t},$$

which, together with (3.5), yields that for any  $f \in L^2(X)$ ,

$$(3.7) \quad f = \eta(s\sqrt{L})f + c_{\Psi,\varphi} \int_0^s \Psi(t\sqrt{L})\varphi(t\sqrt{L})f \frac{dt}{t}.$$

Let  $0 < \theta < 1$ ,  $N\theta > 2n$ . By (3.7), there holds

$$\begin{aligned} \frac{|\varphi(s\sqrt{L})f(y)|}{(1 + \frac{d(x,y)}{s})^N} &\leq \frac{|\eta(s\sqrt{L})\varphi(s\sqrt{L})f(y)|}{(1 + \frac{d(x,y)}{s})^N} + \frac{c_{\Psi,\varphi}}{(1 + \frac{d(x,y)}{s})^N} \left| \int_0^s \varphi(s\sqrt{L})\Psi(t\sqrt{L})\varphi(t\sqrt{L})f(y) \frac{dt}{t} \right| \\ &=: I + II. \end{aligned}$$

Now we apply an argument of Strömberg and Torchinsky as [22, Chapter V, Theorem 5] on page 64. For the term  $I$ , we use (i) of Lemma 2.1 to obtain

$$\begin{aligned} I &\leq \frac{C}{(1 + \frac{d(x,y)}{s})^N} \int_X \frac{1}{V(z,s)} \frac{1}{(1 + \frac{d(y,z)}{s})^N} |\varphi(s\sqrt{L})f(z)| d\mu(z) \\ &\leq C \int_X \frac{1}{V(z,s)} \frac{1}{(1 + \frac{d(x,z)}{s})^N} |\varphi(s\sqrt{L})f(z)| d\mu(z) \end{aligned}$$



$$\leq C \int_X \frac{1}{V(x, s)} \frac{|\varphi(s\sqrt{L})f(z)|^\theta}{\left(1 + \frac{d(x, z)}{s}\right)^{N\theta-n}} d\mu(z) \left(M_{L, \varphi, N}^{**}(f)(x)\right)^{1-\theta},$$

where in the last inequality above we have used (1.3). By noting that  $N\theta > 2n$ , we obtain,

$$(3.8) \quad I \leq C \mathcal{M}(|\varphi_L^+ f|^\theta)(x) \left(M_{L, \varphi, N}^{**}(f)(x)\right)^{1-\theta}.$$

Let us estimate the term  $II$ . One writes  $\Psi(tx)\psi(sx) = (\frac{t}{s})^{2\kappa} [\Phi(tx)(sx)^{2\kappa}\psi(sx)]$ . We then apply an argument as in (ii) of Lemma 2.1 to know that the kernel  $K_{\Psi(t\sqrt{L})\varphi(s\sqrt{L})}(x, y)$  of  $\Psi(t\sqrt{L})\varphi(s\sqrt{L})$  satisfies

$$|K_{\Psi(t\sqrt{L})\varphi(s\sqrt{L})}(y, z)| \leq C \left(\frac{t}{s}\right)^{2\kappa} \frac{1}{V(z, s)} \frac{1}{\left(1 + \frac{d(y, z)}{s}\right)^N}$$

for all  $s > t > 0$  and  $y, z \in X$ . This yields

$$\begin{aligned} II &\leq \frac{C}{\left(1 + \frac{d(x, y)}{s}\right)^N} \int_0^s \int_X \left(\frac{t}{s}\right)^{2\kappa} \frac{1}{V(z, s)} \frac{1}{\left(1 + \frac{d(y, z)}{s}\right)^N} |\varphi(t\sqrt{L})f(z)| d\mu(z) \frac{dt}{t} \\ &\leq C \int_0^s \int_X \left(\frac{t}{s}\right)^{2\kappa} \frac{1}{V(z, s)} \frac{1}{\left(1 + \frac{d(x, z)}{s}\right)^N} |\varphi(t\sqrt{L})f(z)| d\mu(z) \frac{dt}{t} \\ &\leq C \int_0^s \int_X \left(\frac{t}{s}\right)^{2\kappa-N} \frac{1}{V(z, t)} \frac{|\varphi_L^+ f(z)|^\theta}{\left(1 + \frac{d(x, z)}{t}\right)^{N\theta}} d\mu(z) \frac{dt}{t} \left(M_{L, \varphi, N}^{**}(f)(x)\right)^{1-\theta} \\ &\leq C \int_0^s \int_X \left(\frac{t}{s}\right)^{2\kappa-N} \frac{1}{V(x, t)} \frac{|\varphi_L^+ f(z)|^\theta}{\left(1 + \frac{d(x, z)}{t}\right)^{N\theta-n}} d\mu(z) \frac{dt}{t} \left(M_{L, \varphi, N}^{**}(f)(x)\right)^{1-\theta}, \end{aligned}$$

where in the last inequality above we have used (1.3). Since  $2\kappa > N$  and  $N\theta > 2n$ , we have

$$(3.9) \quad \begin{aligned} II &\leq C \int_0^s \left(\frac{t}{s}\right)^{2\kappa-N} \frac{dt}{t} \mathcal{M}(|\varphi_L^+ f|^\theta)(x) \left(M_{L, \varphi, N}^{**}(f)(x)\right)^{1-\theta} \\ &\leq C \mathcal{M}(|\varphi_L^+ f|^\theta)(x) \left(M_{L, \varphi, N}^{**}(f)(x)\right)^{1-\theta}. \end{aligned}$$

Combining (3.8) and (3.9), we have proved that

$$(3.10) \quad M_{L, \varphi, N}^{**}(f)(x) \leq C \mathcal{M}(|\varphi_L^+ f(z)|^\theta)(x) \left(M_{L, \varphi, N}^{**}(f)(x)\right)^{1-\theta}.$$

Finally, it can be verified that for any  $f \in L^2(X)$ ,  $M_{L, \varphi, N}^{**}(f)(x) < \infty$ , for a.e.  $x \in X$ . From (3.10), (3.4) follows readily. The proof of Theorem 3.1 is complete.  $\square$

#### 4. PROOF OF THEOREM 1.3

In this section we continue to show (ii) of Theorem 1.3 to give a  $(p, \infty, M)$ -atomic representation for the Hardy spaces  $H_{L, \max}^p(X)$ . To do it, we first recall the following Whitney type covering lemma on space of homogeneous type  $X$ .

**Lemma 4.1.** *Suppose that  $O \subseteq X$  is a open set with finite measure. There exists a sequence of points  $\{\xi_k\}_{k=1}^\infty \in O$  and a collection of balls  $B(\xi_k, \rho_k)$  where  $\rho_k := d(\xi_k, O^c)$  such that*

- i)  $\bigcup_k B(\xi_k, \rho_k/2) = O$ ;
- ii)  $\{B(\xi_k, \rho_k/10)\}_{k=1}^\infty$  are disjoint.

*Proof.* The proof of this lemma is essentially given in [9, Chapter III, Theorem 1.3] and is omitted. See also [10, 20].  $\square$

*Proof of (ii) of Theorem 1.3.* It suffices to show that for  $f \in H_{L, \max}^p(X) \cap L^2(X)$ ,  $f$  has a  $(p, \infty, M)$  atomic representation.

We start with a suitable version of the Calderón reproducing formula. Let  $\varphi \in C_0^\infty(\mathbb{R})$  be an even function with  $\text{supp } \varphi \subset (-1, 1)$ . Let  $\Phi$  denote the Fourier transform of  $\varphi$ , and set  $\Psi(x) := x^{2M}\Phi(x)$ ,  $x \in \mathbb{R}$ . By the spectral theory ([25]), for every  $f \in L^2(X)$  one can write

$$(4.1) \quad f = \lim_{\epsilon \rightarrow 0} c_\Psi \int_\epsilon^{1/\epsilon} \Psi(t\sqrt{L}) t^2 L e^{-t^2 L} f \frac{dt}{t}$$

with the integral converging in  $L^2(X)$ .

Set

$$\eta(x) := c_\Psi \int_1^\infty t^2 x^2 \Psi(tx) e^{-t^2 x^2} \frac{dt}{t} = c_\Psi \int_x^\infty y \Psi(y) e^{-y^2} dy, \quad x \neq 0$$

with  $\eta(0) = 1$ . Then  $\eta \in \mathcal{S}(\mathbb{R})$  is an even function. By the spectral theory ([25]) again, one has

$$(4.2) \quad c_\Psi \int_a^b \Psi(t\sqrt{L}) t^2 L e^{-t^2 L} f \frac{dt}{t} = \eta(a\sqrt{L})f(x) - \eta(b\sqrt{L})f(x).$$

Define,

$$\mathcal{M}_L f(x) := \sup_{d(x,y) < 5t} (|t^2 L e^{-t^2 L} f(y)| + |\eta(t\sqrt{L})f(y)|).$$

By Proposition 2.3, it follows that

$$\|\mathcal{M}_L f\|_{L^p(X)} \leq C \|f\|_{H_{L, \max}^p(X)}, \quad 0 < p \leq 1.$$

In the sequel, if  $O$  is an open subset of  $\mathbb{R}^n$ , then the “tent” over  $O$ , denoted by  $\widehat{O}$ , is given as  $\widehat{O} := \{(x, t) \in X \times (0, +\infty) : B(x, 4t) \subset O\}$ . For  $i \in \mathbb{Z}$ , we define the family of sets  $O_i := \{x \in X : \mathcal{M}_L f(x) > 2^i\}$ . We obtain a decomposition for  $X \times (0, +\infty)$  as follows:

$$(4.3) \quad \begin{aligned} X \times (0, +\infty) &= \bigcup_{i \in \mathbb{Z}} \widehat{O}_i \\ &= \bigcup_{i \in \mathbb{Z}} (\widehat{O}_i \setminus \widehat{O}_{i+1}) \\ &=: \bigcup_{i \in \mathbb{Z}} T_i, \end{aligned}$$

where

$$T_i := \widehat{O}_i \setminus \widehat{O}_{i+1}.$$

Note that for each  $i \in \mathbb{Z}$ ,  $O_i$  is open set with  $\mu(O_i) < \infty$ . By Lemma 4.1, we can further decomposition  $O_i$  into “balls” of  $X$ . More precisely, for each  $i \in \mathbb{Z}$ , there exists a sequence of points  $\{\xi_i^k\}_{k=1}^\infty \in O_i$ , such that

- 1)  $O_i = \bigcup_{k=1}^\infty B_i^k$ ;
- 2)  $\{\frac{1}{5}B_i^k\}_{k=1}^\infty$  are disjoint, where  $B_i^k := B(\xi_i^k, \rho_i^k/2)$  and  $\rho_i^k := d(\xi_i^k, O_i^c)$ .

For any  $E \subset X$ , we define the “cone” of  $E$  by

$$(4.4) \quad \mathcal{R}(E) := \{(y, t) : d(y, E) < 2t\}.$$

For every  $k = 0, 1, 2, \dots$ , we set

$$(4.5) \quad \mathcal{R}(B_i^0) := \emptyset, \quad T_i^k := T_i \cap (\mathcal{R}(B_i^k) \setminus \cup_{j=0}^{k-1} \mathcal{R}(B_i^j)).$$

It is easy to see that  $\widehat{O}_i \subset \cup_{j \in \mathbb{N}} \mathcal{R}(B_i^j)$  and  $T_i^k \cap T_{i'}^{k'} = \emptyset$  if  $k \neq k'$  or  $i \neq i'$ . By (4.3) and (4.5), we can obtain a further decomposition for  $X \times (0, +\infty)$  as follows:

$$(4.6) \quad \begin{aligned} X \times (0, +\infty) &= \bigcup_{i \in \mathbb{Z}} \bigcup_{k \in \mathbb{N}} (T_i \cap \mathcal{R}(B_i^k)) \\ &= \bigcup_{i \in \mathbb{Z}} \bigcup_{k \in \mathbb{N}} (T_i \cap (\mathcal{R}(B_i^k) \setminus \cup_{j=0}^{k-1} \mathcal{R}(B_i^j))) \\ &= \bigcup_{i \in \mathbb{Z}} \bigcup_{k \in \mathbb{N}} T_i^k. \end{aligned}$$

By (4.1), this leads us to write

$$(4.7) \quad \begin{aligned} f &= \sum_{i \in \mathbb{Z}, k \in \mathbb{N}} c_\Psi \int_0^\infty \Psi(t \sqrt{L}) (\chi_{T_i^k} t^2 L e^{-t^2 L} f) \frac{dt}{t} \\ &=: \sum_{i \in \mathbb{Z}, k \in \mathbb{N}} \lambda_i^k a_i^k, \end{aligned}$$

where  $\lambda_i^k := 2^i \mu(B_i^k)^{1/p}$ ,  $a_i^k := L^M b_i^k$ , and

$$b_i^k := (\lambda_i^k)^{-1} c_\Psi \int_0^\infty t^{2M} \Phi(t \sqrt{L}) (\chi_{T_i^k} t^2 L e^{-t^2 L} f) \frac{dt}{t}.$$

We see that the sum (4.7) converges in  $L^2(X)$ . Indeed, since for each  $f \in L^2(X)$ ,

$$\left( \int_{X \times (0, +\infty)} |t^2 L e^{-t \sqrt{L}} f(y)|^2 \frac{d\mu(y) dt}{t} \right)^{1/2} \leq C \|f\|_{L^2(X)}.$$

By (4.7),

$$\begin{aligned} \left\| \sum_{|i| > N_1, k > N_2} \lambda_i^k a_i^k \right\|_{L^2(X)} &= c_\Psi \left\| \sum_{|i| > N_1, k > N_2} \int_{X \times (0, +\infty)} K_{(t^2 L)^M \Phi(t \sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t \sqrt{L}} f(y) \frac{d\mu(y) dt}{t} \right\|_{L^2(X)} \\ &\leq \sup_{\|g\|_2 \leq 1} \sum_{|i| > N_1, k > N_2} \int_{T_i^k} |(t^2 L)^M \Phi(t \sqrt{L}) g(y) t^2 L e^{-t \sqrt{L}} f(y)| \frac{d\mu(y) dt}{t} \\ &\leq C \left( \sum_{|i| > N_1, k > N_2} \int_{T_i^k} |t^2 L e^{-t \sqrt{L}} f(y)|^2 \frac{d\mu(y) dt}{t} \right)^{1/2} \rightarrow 0 \end{aligned}$$

as  $N_1 \rightarrow \infty, N_2 \rightarrow \infty$ .

Next, we will show that, up to a normalization by a multiplicative constant, the  $a_i^k$  are  $(p, \infty, M)$ -atoms. Once the claim is established, we have

$$\begin{aligned} \sum_{i \in \mathbb{Z}, k \in \mathbb{N}} |\lambda_i^k|^p &= \sum_{i \in \mathbb{Z}, k \in \mathbb{N}} 2^{ip} \mu(B_i^k) \leq 5^n \sum_{i \in \mathbb{Z}, k \in \mathbb{N}} 2^{ip} \mu\left(\frac{1}{5} B_i^k\right) \\ &\leq C \sum_{i \in \mathbb{Z}} 2^{ip} \mu(O_i) \end{aligned}$$

$$\leq C \|f\|_{H_{L,max}^p(X)}^p$$

as desired.

Let us now prove that for every  $i \in \mathbb{Z}$  and  $k \in \mathbb{N}$ , the function  $C^{-1}d_i^k$  is a  $(p, \infty, M)$ -atom associated with the ball  $B(\xi_i^k, 5\rho_i^k)$  for some constant  $C$ . Observe that if  $(y, t) \in T_i^k$ , then  $B(y, 4t) \in O_i$ . It implies that  $4t \leq d(y, (O_i)^c)$ . Note that  $d(y, B_i^k) < 2t$ , then  $d(y, (O_i)^c) \leq d(y, B_i^k) + 2\rho_i^k < 2t + 2\rho_i^k$ . Hence, we have that  $t < \rho_i^k$ . From the formula (2.3), it is easy to see that the integral kernel  $K_{(t^2L)^k\Phi(t\sqrt{L})}$  of the operator  $(t^2L)^k\Phi(t\sqrt{L})$  satisfies

$$\text{supp } K_{(t^2L)^k\Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}.$$

It follows that  $d(x, B_i^k) \leq d(x, y) + d(y, B_i^k) < 3t \leq 3\rho_i^k$  and  $d(x, \xi_i^k) < 4\rho_i^k$ . Hence, for every  $j = 0, 1, \dots, M$

$$\text{supp } (L^j b_i^k) \subseteq B(\xi_i^k, 4\rho_i^k).$$

It remains to show that  $\|(\rho_i^k)^{-2}L^j b_i^k\|_{L^\infty(X)} \leq C(\rho_i^k)^{2M}\mu(B_i^k)^{-1/p}$ ,  $j = 0, 1, \dots, M$ .

In this case  $j = 0, 1, \dots, M-1$ , it reduces to show

$$(4.8) \quad \left| \int_0^\infty \int_{\mathbb{R}^n} K_{t^{2M}L^j\Phi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2L} f(y) d\mu(y) \frac{dt}{t} \right| \leq C 2^i (\rho_i^k)^{2(M-j)}.$$

Indeed, if  $\chi_{T_i^k}(y, t) = 1$ , then  $(y, t) \in (\widehat{O_{i+1}})^c$ , and so  $B(y, 4t) \cap (O_{i+1})^c \neq \emptyset$ . Let  $\bar{x} \in B(y, 4t) \cap (O_{i+1})^c$ . We have that  $|t^2 L e^{-t^2L} f(y)| \leq \mathcal{M}_L f(\bar{x}) \leq 2^{i+1}$ . By (i) of Lemma 2.1,

$$\begin{aligned} & \left| \int_0^\infty \int_{\mathbb{R}^n} K_{t^{2M}L^j\Phi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2L} f(y) d\mu(y) \frac{dt}{t} \right| \\ & \leq C 2^i \left| \int_0^{\rho_i^k} t^{2(M-j)} \int_{\mathbb{R}^n} |K_{(t^2L)^j\Phi(t\sqrt{L})}(x, y)| d\mu(y) \frac{dt}{t} \right| \\ & \leq C 2^i \int_0^{\rho_i^k} t^{2(M-j)} \frac{dt}{t} \\ & \leq C 2^i (\rho_i^k)^{2(M-j)} \end{aligned}$$

since  $j = 0, 1, \dots, M-1$ .

Next, let us consider this case  $j = M$ . we will show that for every  $i \in \mathbb{Z}, k \in \mathbb{N}$ ,

$$(4.9) \quad \left| \int_0^\infty \int_X K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2L} f(y) d\mu(y) \frac{dt}{t} \right| \leq C 2^i.$$

To prove (4.9), we need the following result, which plays a crucial role in the proof of (ii) of Theorem 1.3.

**Lemma 4.2.** *Fix  $x \in O_i$ , the properties of the set defining  $\chi_{T_i^k}(y, t)$  imply that there exist intervals  $(0, b_0), (a_1, b_1), \dots, (a_N, +\infty)$ , where  $0 < b_0 \leq a_1 < b_1 \leq \dots \leq a_N < +\infty$ ,  $1 \leq N \leq 4$  such that for  $j = 0, 1, \dots, N-1$ , there hold  $a_{j+1} \leq 3^4 b_j$  and*

- (a)  $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) = 0$  for  $t > a_N$ ;
- (b) either  $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) = 0$  or  $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) = K_{\Psi(t\sqrt{L})}(x, y)$  for all  $t \in (a_j, b_j)$ ;
- (c) either  $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) = 0$  or  $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) = K_{\Psi(t\sqrt{L})}(x, y)$  for all  $t \in (0, b_0)$ .

*Proof.* Recall that for any set  $E \subset X$ ,  $\mathcal{R}(E)$  is given in (4.4). It is easy to see that for every set  $E_1 \subset X$  and  $E_2 \subset X$ , there holds  $\mathcal{R}(E_1) \cup \mathcal{R}(E_2) = \mathcal{R}(E_1 \cup E_2)$ . One can write

$$\begin{aligned} \mathcal{R}(B_i^k) \setminus \bigcup_{j=0}^{k-1} \mathcal{R}(B_i^j) &= \bigcup_{j=0}^k \mathcal{R}(B_i^j) \setminus \bigcup_{j=0}^{k-1} \mathcal{R}(B_i^j) \\ &= \mathcal{R}\left(\bigcup_{j=0}^k B_i^j\right) \setminus \mathcal{R}\left(\bigcup_{j=0}^{k-1} B_i^j\right) \\ &:= \mathcal{R}(E_i^k) \setminus \mathcal{R}(E_i^{k-1}), \end{aligned}$$

which, in combination with  $T_i^k = T_i \cap (\mathcal{R}(B_i^k) \setminus \bigcup_{j=1}^{k-1} \mathcal{R}(B_i^j))$  (see (4.5)), gives

$$\begin{aligned} \chi_{T_i^k}(y, t) &= \chi_{\widehat{O_i}}(y, t) \cdot \chi_{(\widehat{O_{i+1}})^c}(y, t) \cdot \chi_{\mathcal{R}(E_i^k)}(y, t) \cdot \chi_{(\mathcal{R}(E_i^{k-1}))^c}(y, t) \\ (4.10) \quad &=: \prod_{\ell=1}^4 \chi_\ell(y, t). \end{aligned}$$

From (4.10), we know that if  $\chi_{T_i^k}(y, t) = 1$ , then  $\chi_\ell(y, t) = 1$  for all  $\ell = 1, 2, 3, 4$ . That is, if either of  $\chi_\ell(y, t) = 0$ , then  $\chi_{T_i^k}(y, t) = 0$ .

To prove Lemma 4.2, we claim that for  $\ell = 1, 2, 3, 4$ , there exist numbers  $b^{(\ell)}$  and  $a^{(\ell+1)}$ ,  $0 < b^{(\ell)} \leq a^{(\ell+1)}$ ,  $a^{(\ell+1)} \leq 3b^{(\ell)}$  such that either  $K_{\Psi(t\sqrt{L})}(x, y)\chi_\ell(y, t) = 0$  or  $K_{\Psi(t\sqrt{L})}(x, y)\chi_\ell(y, t) = K_{\Psi(t\sqrt{L})}(x, y)$  for all  $t$  in each of the intervals complementary to  $(b^{(\ell)}, a^{(\ell+1)})$ . And for at least one of  $\chi_\ell(y, t)$ ,  $K_{\Psi(t\sqrt{L})}(x, y)\chi_\ell(y, t) = 0$  for  $\ell > a^{(\ell+1)}$ . By (4.10), we see that

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{T_{ij}}(y, t) = \prod_{\ell=1}^4 \chi_\ell(y, t) K_{\Psi(t\sqrt{L})}(x, y)$$

equals  $K_{\Psi(t\sqrt{L})}(x, y)$  or 0 when  $t$  is in each of the intervals complementary to  $\bigcup_{\ell=1}^4 (b^{(\ell)}, a^{(\ell+1)})$ . From this, Lemma 4.2 follows readily.

We now prove our claim. Fix  $x \in O_i$  and  $d(x, y) < t$ . Let us consider the following four cases.

**Case 1:**  $\chi_1(y, t) = \chi_{\widehat{O_i}}(y, t) = 1$ .

In this case, we choose  $b^{(1)} = \frac{1}{5}d(x, O_i^c)$  and  $a^{(2)} = \frac{1}{2}d(x, O_i^c)$ , and so  $a^{(2)} \leq 3b^{(1)}$ . If  $t < b^{(1)}$ , then  $d(y, O_i^c) \geq d(x, O_i^c) - d(x, y) > 5t - t = 4t$ . This tells us

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{\widehat{O_i}}(y, t) = K_{\Psi(t\sqrt{L})}(x, y), \quad \text{for } t < b^{(1)}.$$

On the other hand, if  $t > a^{(2)}$ , then  $d(y, O_i^c) \leq d(x, O_i^c) + d(x, y) < 4t$ . From this, we have

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{\widehat{O_i}}(y, t) = 0, \quad \text{for } t > a^{(2)}.$$

**Case 2:**  $\chi_2(y, t) = \chi_{(\widehat{O_{i+1}})^c}(y, t) = 1$ .

In this case, we consider two cases:  $d(x, O_{i+1}^c) = 0$  and  $d(x, O_{i+1}^c) > 0$ .

**Subcase 2.1:**  $d(x, O_{i+1}^c) = 0$ .

It follows that  $d(y, O_{i+1}^c) \leq d(x, O_{i+1}^c) + d(x, y) < t < 4t$ . Hence,

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{(\widehat{O_{i+1}})^c}(y, t) = K_{\Psi(t\sqrt{L})}(x, y), \quad \text{for } t > 0.$$

So we can choose  $b^{(2)}$  and  $a^{(3)}$  to be any positive number. For example, we let  $b^{(2)} = b^{(1)}$  and  $a^{(3)} = a^{(2)}$ .

**Subcase 2.2:**  $d(x, O_{i+1}^c) > 0$ .

Let us choose  $b^{(2)} = \frac{1}{5}d(x, O_{i+1}^c)$  and  $a^{(3)} = \frac{1}{2}d(x, O_{i+1}^c)$ . If  $t < b^{(2)}$ , then  $d(y, O_{i+1}^c) \geq d(x, O_{i+1}^c) - d(x, y) > 5t - t = 4t$ . which gives

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{(\widehat{O_{i+1}^c})^c}(y, t) = 0 \quad \text{for } t < b^{(2)},$$

If  $t > a^{(3)}$ , then  $d(y, O_{i+1}^c) \leq d(x, O_{i+1}^c) + d(x, y) < 4t$ . Therefore,

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{(\widehat{O_{i+1}^c})^c}(y, t) = K_{\Psi(t\sqrt{L})}(x, y), \quad \text{for } t > a^{(3)}.$$

**Case 3:**  $\chi_3(y, t) = \chi_{\mathcal{R}(E_i^k)}(y, t) = 1$ .

In this case, we consider two cases:  $d(x, E_i^k) = 0$  and  $d(x, E_i^k) > 0$ .

**Subcase 3.1:**  $d(x, E_i^k) = 0$ .

It follows that  $d(y, E_i^k) \leq d(x, y) < t < 2t$ . Hence,

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{\mathcal{R}(E_i^k)}(y, t) = K_{\Psi(t\sqrt{L})}(x, y), \quad \text{for } t > 0.$$

In this case, we can choose  $b^{(3)}$  and  $a^{(4)}$  to be any positive number. For example, we let  $b^{(3)} = b^{(1)}$  and  $a^{(4)} = a^{(2)}$ .

**Subcase 3.2:**  $d(x, E_i^k) > 0$ .

We choose  $b^{(3)} = d(x, E_i^k)/3$  and  $a^{(4)} = d(x, E_i^k)$ . If  $t < b^{(3)}$ , then  $d(y, E_i^k) \geq d(x, E_i^k) - d(x, y) > 3t - t = 2t$ . Hence,

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{\mathcal{R}(E_i^k)}(y, t) = 0 \quad \text{for } t < b^{(3)}.$$

If  $t > a^{(4)}$ , then  $d(y, E_i^k) \leq d(x, E_i^k) + d(x, y) < 2t$ . This tells us that for  $t > a^{(4)}$ ,

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{\mathcal{R}(E_i^k)}(y, t) = K_{\Psi(t\sqrt{L})}(x, y).$$

**Case 4:**  $\chi_4(y, t) = \chi_{(\mathcal{R}(E_i^{k-1}))^c}(y, t) = 1$ .

In this case, we consider two cases:  $d(x, E_i^{k-1}) = 0$  and  $d(x, E_i^{k-1}) > 0$ .

**Subcase 4.1:**  $d(x, E_i^{k-1}) = 0$ .

It follows that  $d(y, E_i^{k-1}) \leq d(x, y) < t < 2t$ . Hence,

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{(\mathcal{R}(E_i^{k-1}))^c}(y, t) = 0, \quad \text{for } t > 0.$$

We let  $b^{(4)}$  and  $a^{(5)}$  be any positive number. For example, we choose  $b^{(4)} = b^{(1)}$  and  $a^{(5)} = a^{(2)}$ .

**Subcase 4.2:**  $d(x, E_i^{k-1}) > 0$ .

We choose  $b^{(4)} = d(x, E_i^{k-1})/3$  and  $a^{(5)} = d(x, E_i^{k-1})$ . If  $t < b^{(4)}$ , then  $d(y, E_i^{k-1}) \geq d(x, E_i^{k-1}) - d(x, y) > 3t - t = 2t$ . This tells us

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{(\mathcal{R}(E_i^{k-1}))^c}(y, t) = K_{\Psi(t\sqrt{L})}(x, y) \quad \text{for } t < b^{(4)}.$$

If  $t > d(x, E_i^{k-1})$ , then  $d(y, E_i^{k-1}) \leq d(x, E_i^{k-1}) + d(x, y) < 2t$ . Therefore,

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_{(\mathcal{R}(E_i^{k-1}))^c}(y, t) = 0, \quad \text{for } t > a^{(5)}.$$

From **Cases 1, 2, 3** and **4** above, we have obtained our claim, and then the proof of Lemma 4.2 is complete.  $\square$

*Back to the proof of (ii) of Theorem 1.3.* We continue to show (4.9). Note that the conditions  $d(x, y) < t$  and  $B(y, 4t) \in O_i$  imply that  $x \in O_i$ . If  $x \in O_i^c$ , then

$$\int_0^\infty \int_X K_{\Psi(t\sqrt{L})}(x, y)\chi_{T_i^k}(y, t)t^2 Le^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} = 0.$$

Fix  $x \in O_i$ . We apply Lemma 4.2 to write

$$\begin{aligned}
 (4.11) \quad & \int_0^\infty \int_X K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} \\
 &= \left\{ \int_0^{b_0} + \sum_{l=1}^{N-1} \int_{a_l}^{b_l} \right\} \int_X K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} \\
 &+ \left\{ \sum_{l=0}^{N-1} \int_{b_l}^{a_{l+1}} \right\} \int_X K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} \\
 &= I_1(x) + I_2(x).
 \end{aligned}$$

To estimate  $I_1(x)$ , we note that if  $0 \leq a < b \leq b_1$  or  $a_l \leq a < b \leq b_l$ , then one has either

$$\int_a^b \int_X K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} = 0,$$

or by (4.2),

$$\begin{aligned}
 \int_a^b \int_X K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} &= \int_a^b \Psi(t\sqrt{L}) t^2 L e^{-t^2 L} f(x) \frac{dt}{t} \\
 &= \eta(a\sqrt{L})f(x) - \eta(b\sqrt{L})f(x).
 \end{aligned}$$

Observe that for each  $a \leq t \leq b$ , if  $d(x, y) < t$ , then  $\chi_{T_i^k}(y, t) = 1$ . This tells us that  $(y, t) \in (\widehat{O_{i+1}})^c$ , hence  $B(y, 4t) \cap (O_{i+1})^c \neq \emptyset$ . Assume that  $\bar{x} \in B(y, 4t) \cap (O_{i+1})^c$ . From this, we have that  $d(x, \bar{x}) \leq d(x, y) + d(y, \bar{x}) < 5t$  and  $\mathcal{M}_L f(\bar{x}) \leq 2^{i+1}$ . It implies that  $|\eta(t\sqrt{L})f(x)| \leq \mathcal{M}_L f(\bar{x}) \leq C2^{i+1}$  for every  $a \leq t \leq b$ . Therefore,  $|\eta(a\sqrt{L})f(x)| \leq C2^{i+1}$  and  $|\eta(b\sqrt{L})f(x)| \leq C2^{i+1}$ , and so  $|I_1(x)| \leq C2^{i+1}$ .

Consider  $I_2(x)$ . If  $\chi_{T_i^k}(y, t) = 1$ , then  $(y, t) \in (\widehat{O_{i+1}})^c$ . Thus  $B(y, 4t) \cap (O_{i+1})^c \neq \emptyset$ . Assume that  $\bar{x} \in B(y, 4t) \cap (O_{i+1})^c$ . We have that  $|t^2 L e^{-t^2 L} f(y)| \leq \mathcal{M}_L f(\bar{x}) \leq 2^{i+1}$ . This, together with  $a_{l+1} \leq 3^4 b_l$  ( $l = 0, 1, \dots, N-1$ ), implies that

$$\begin{aligned}
 (4.12) \quad & \left| \int_{b_l}^{a_{l+1}} \int_X K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_i^k}(y, t) t^2 L e^{-t^2 L} f(y) d\mu(y) \frac{dt}{t} \right| \leq 2^{i+1} \left| \int_{b_l}^{81b_l} \int_X |K_{\Psi(t\sqrt{L})}(x, y)| d\mu(y) \frac{dt}{t} \right| \\
 & \leq C2^{i+1} \int_{b_l}^{81b_l} \frac{1}{t} dt \leq C2^{i+1},
 \end{aligned}$$

which yields that  $|I_2(x)| \leq C2^{i+1}$ .

Combining estimates of  $I_1(x)$  and  $I_2(x)$ , we have obtained (4.9). This proves (ii) of Theorem 1.3, and the proof of Theorem 1.3 is complete.  $\square$

**Remarks.** i) In [10], the authors established the equivalence of the maximal and atomic Hardy spaces associated to an operator  $L$  on the space of homogeneous type  $X$ , under the following four assumptions that  $L$  satisfies **(H1)**-**(H2)**, and

**(H3)** the kernel  $p_t(x, y)$  of the semigroup  $e^{-tL}$  satisfies the Hölder continuity: There exists a constant  $\alpha > 0$  such that

$$|p_t(x, y) - p_t(x, y')| \leq C \left( \frac{d(y, y')}{\sqrt{t}} \right)^\alpha$$

for  $x, y, y' \in X$  and  $t > 0$ , whenever  $d(y, y') \leq \sqrt{t}$ ; and

**(H4)** Markov property:

$$\int_X p_t(x, y) d\mu(y) = 1$$



for  $x \in X$  and  $t > 0$ .

ii) Let  $L$  be an operator satisfying (H1)-(H2). For  $f \in L^2(X)$ , we define an area function  $S_L f$  associated to the heat semigroup generated by  $L$ ,

$$(4.13) \quad S_L f(X) := \left( \int_0^\infty \int_{d(x,y) < t} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y) dt}{tV(y, t)} \right)^{1/2}, \quad x \in X.$$

Given  $0 < p \leq 1$ . The Hardy space  $H_{L,S}^p(\mathbb{R}^n)$  is defined as the completion of  $\{f \in L^2(X) : \|S_L f\|_{L^p(X)} < \infty\}$  with norm

$$\|f\|_{H_{L,S}^p(X)} := \|S_L f\|_{L^p(X)}.$$

From [11], [15] and Theorem 1.3 above,

$$H_{L,at,q,M}^p(X) \simeq H_{L,S}^p(X) \simeq H_{L,max}^p(X) \simeq H_{L,rad}^p(X)$$

for every  $0 < p \leq 1$ , and for all  $q > p$  with  $1 \leq q \leq \infty$  and all integers  $M > \frac{n}{2}(\frac{1}{p} - 1)$ .

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